

P-PARTITION PRODUCTS AND FUNDAMENTAL QUASI-SYMMETRIC FUNCTION POSITIVITY

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ABSTRACT. We show that certain differences of products

$$K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}$$

of P -partition generating functions are positive in the basis of fundamental quasi-symmetric functions L_α . This result interpolates between recent Schur positivity and monomial positivity results of the same flavor. We study the case of chains in detail, introducing certain “cell transfer” operations on compositions and an interesting related “ L -positivity” poset. We introduce and study quasi-symmetric functions called *wave Schur functions* and use them to establish, in the case of chains, that $K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}$ is itself equal to a single generating function $K_{P, \theta}$ for a labeled poset (P, θ) . In the course of our investigations we establish some factorization properties of the ring QSym of quasisymmetric functions.

1. INTRODUCTION

The present article studies quasi-symmetric functions $f \in \text{QSym}$ which are non-negative linear combinations of fundamental quasi-symmetric functions. It sits between joint work with Postnikov [LPP] involving symmetric functions and Schur positivity and our more poset-theoretic work [LP] involving monomial positivity.

The Schur functions s_λ (see [Sta99]) are important symmetric functions which occur in combinatorics, representation theory and geometry. They exhibit a multitude of remarkable properties, and here we highlight three non-trivial *positivity* properties.

- (A) The product $s_\lambda s_\mu$ of two Schur functions is Schur-positive (Littlewood-Richardson rule I).
- (B) A skew Schur function $s_{\lambda/\mu}$ is Schur-positive (Littlewood-Richardson rule II).
- (C) The difference of products $s_{\max(\lambda/\mu, \nu/\rho)} s_{\min(\lambda/\mu, \nu/\rho)} - s_{\lambda/\mu} s_{\nu/\rho}$ is Schur positive (Lam-Postnikov-Pylyavskyy [LPP]). Here max and min are taken coordinate-wise.

The aim of this article is to replace symmetric functions with quasi-symmetric functions and study analogous positivity properties for the fundamental quasi-symmetric functions L_α . Say that a quasi-symmetric function f is *L -positive* if it is a non-negative linear combination of fundamental quasi-symmetric functions. We take the point of view that the quasi-symmetric analogues of (A) and (B) are:

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(A*) The product $L_\alpha L_\beta$ of fundamental quasi-symmetric functions is L -positive (shuffle product).

(B*) For any poset P and labeling $\theta : P \rightarrow \mathbb{P}$ the generating function $K_{P,\theta}$ of P -partitions is L -positive (Stanley’s P -partition theory [Sta72]).

Thus the functions $K_{P,\theta}$ will replace the skew Schur functions $s_{\lambda/\mu}$. Our main result (Theorem 4.2), which is the analogue of (C), says that the difference

$$(1) \quad K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}$$

is L -positive, where Q and R are two convex subsets of a labeled poset (P, θ) and \wedge and \vee are the *cell transfer operations* introduced in [LP].

In [LP], the same difference (1) is shown to be monomial-positive for a larger class of posets called \mathbb{T} -labeled posets. Since Schur-positivity implies L -positivity which in turn implies monomial-positivity, our current result sits between the two results of [LP] and [LPP]; with each restriction to the class of (labeled) posets a stronger form of positivity holds. We summarize the relationship between this article and the two previous works [LP, LPP] in a table.

Paper	Cell Transfer [LP]	This	Schur positivity [LPP]
Ring	$\mathbb{Z}[[x_1, x_2, \dots]]$	QSym	Sym
Basis	x^α	L_α	s_λ
Skew fcns.	$K_{P,O}$	$K_{P,\theta}$	$s_{\lambda/\mu}$
Posets	\mathbb{T} -labeled posets (P, O)	Stanley’s (P, θ)	Young diagrams λ/μ

In each of the three cases, the structure constants in the “basis” are non-negative, and the “skew functions” lie in the “ring” and are non-negative when written in terms of the “basis”. In all three cases, the difference of products of “skew functions” arising from the cell transfer operation on the “posets” is positive in the corresponding “basis”. The Schur functions have an interpretation as irreducible characters of the symmetric group while the fundamental quasi-symmetric functions have an interpretation as irreducible characters of the 0-Hecke algebra. It would be interesting to give a representation theoretic explanation of our results (and in particular of the cell transfer operations).

We study the difference (1) in detail for the special case that Q and R are convex subsets of a chain, in which case all the four functions in this difference are themselves fundamental quasi-symmetric functions L_α . We introduce “cell transfer” operations on compositions, also denoted \vee and \wedge , such that the difference $L_{\alpha \wedge \beta} L_{\alpha \vee \beta} - L_\alpha L_\beta$ is L -positive. We further conjecture (Conjecture 5.5) that a product $L_\alpha L_\beta$ is maximal in “ L -positivity order” if and only if the pair $\{\alpha, \beta\}$ is stable under cell transfer. As part of this investigation, we show that each L_α is irreducible.

Next, we ask when the difference (1) is itself equal to $K_{P,\theta}$ for some labeled poset (P, θ) . We show that this is always the case for the differences $L_{\alpha \wedge \beta} L_{\alpha \vee \beta} - L_\alpha L_\beta$ by introducing generating functions we call *wave Schur functions*, which appear to be interesting in their own right.

Wave Schur functions are generating functions of certain Young tableaux, where the weakly and strictly increasing conditions are altered in a particular alternating pattern. We call these tableaux “*wave p-tableaux*” where \mathbf{p} indicates how the increasing conditions have been modified. We show that wave Schur functions are L -positive and that they satisfy a Jacobi-Trudi style determinantal formula

(Theorem 6.6), with the fundamental quasi-symmetric functions replacing the homogeneous symmetric functions. The difference $L_{\alpha \wedge \beta} L_{\alpha \vee \beta} - L_{\alpha} L_{\beta}$ is equal to an appropriate wave Schur function for a two-row skew shape.

In the final sections of the paper, we comment on whether our results can be expanded to a larger class of generating functions of the form $K_{P,O}$ for a \mathbb{T} -labeled poset (P, O) .

2. QUASI-SYMMETRIC FUNCTIONS

We refer to [Sta99] for more details of the material in this section.

2.1. Monomial and fundamental quasi-symmetric functions. Let n be a positive integer. A *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of positive integers such that $\alpha_1 + \alpha_2 + \dots + \alpha_k = n$. We write $|\alpha| = n$. Denote the set of compositions of n by $\text{Comp}(n)$. Associated to a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of n is a subset $D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \dots + \alpha_{k-1}\}$ of $[n-1]$. The map $\alpha \mapsto D(\alpha)$ is a bijection between compositions of n and subsets of $[n-1] = \{1, 2, \dots, n-1\}$. We will denote the inverse map by $\mathcal{C} : 2^{[n-1]} \rightarrow \text{Comp}(n)$ so that $\mathcal{C}(D(\alpha)) = \alpha$.

A formal power series $f = f(x) \in \mathbb{Z}[[x_1, x_2, \dots]]$ with bounded degree is called *quasi-symmetric* if for any $a_1, a_2, \dots, a_k \in \mathbb{P}$ we have

$$[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}] f = [x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}] f$$

whenever $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. Here $[x^\alpha]f$ denotes the coefficient of x^α in f . Denote by $\text{QSym} \subset \mathbb{Z}[[x_1, x_2, \dots]]$ the ring of quasi-symmetric functions.

Let α be a composition. Then the *monomial quasi-symmetric function* M_α is given by

$$M_\alpha = \sum_{i_1 < \dots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}.$$

The *fundamental quasi-symmetric function* L_α is given by

$$L_\alpha = \sum_{D(\beta) \subset D(\alpha)} M_\beta,$$

where the summation is over compositions β satisfying $|\beta| = |\alpha|$. The set of fundamental quasi-symmetric functions (resp. monomial quasi-symmetric functions) form a basis of QSym . We say that a quasi-symmetric function $f \in \text{QSym}$ is *L-positive* (resp. *M-positive*) if it is a non-negative linear combination of fundamental quasi-symmetric functions (resp. monomial quasi-symmetric functions). Note that *L-positivity* implies *M-positivity*.

Two fundamental quasi-symmetric functions L_α and L_β multiply according to the *shuffle product*. Let $u = u_1 u_2 \cdots u_k$ and $v = v_1 v_2 \cdots v_l$ be two words. Then a word $w = w_1 w_2 \cdots w_{k+l}$ is a shuffle of u and v if there exist disjoint subsets $A, B \subset [k+l]$ such that $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_l\}$, $w_{a_i} = u_i$ for all $1 \leq i \leq k$, $w_{b_i} = v_i$ for all $1 \leq i \leq l$ and $A \cup B = [k+l]$. We denote the set of shuffles of u and v by $u \odot v$.

For a composition α with $|\alpha| = n$ let $w(\alpha) = w = w_1 w_2 \cdots w_n$ denote any word with *descent set* $D(w) = \{i : w_i > w_{i+1}\} \subset [n-1]$ equal to $D(\alpha)$. Suppose $w(\alpha)$

and $w(\beta)$ are chosen to have disjoint letters. Then

$$L_\alpha L_\beta = \sum_{u \in w(\alpha) \odot w(\beta)} L_{\mathcal{C}(u)},$$

where $\mathcal{C}(u)$ is by definition the composition $\mathcal{C}(D(u))$ associated to u .

2.2. Two involutions on QSym. If $D \subset [n-1]$ we let $\bar{D} = \{i \in [n-1] \mid i \notin D\}$ denote its complement. For a composition α , define $\bar{\alpha} = \mathcal{C}(\overline{D(\alpha)})$. Let ω denote the linear endomorphism of QSym given by $\omega(L_\alpha) = L_{\bar{\alpha}}$.

Let α^* denote α read backwards: $\alpha^* = (\alpha_k, \dots, \alpha_1)$. Let ν be the linear endomorphism of QSym which sends $L_\alpha \mapsto L_{\alpha^*}$.

Proposition 2.1. *The maps ω and ν are algebra involutions of QSym, and we have $\nu(M_\alpha) = M_{\alpha^*}$.*

Proof. We will check the first statement for ν ; the proof for ω is similar. Let $w = w_1 w_2 \cdots w_r \in S_r$ be a permutation with descent set $D(w) = D(\alpha)$. Then $w^* \in S_r$ given by $w^* = (r+1-w_r)(r+1-w_{r-1}) \cdots (r+1-w_1)$ has descent set $D(w^*) = D(\alpha^*)$. If $u \in S_{r+l}$ is a shuffle of $w \in S_r$ and $v \in S_l$, where $v \in S_l$ uses the letters $r+1, r+2, \dots, r+l$, then u^* is a shuffle of $v^* \in S_l$ and $w^* \in S_r$ where $w^* \in S_r$ uses the letters $l+1, l+2, \dots, r+l$. Thus $\nu(L_{\mathcal{C}(v)}) \nu(L_{\mathcal{C}(w)}) = L_{\mathcal{C}(v)^*} L_{\mathcal{C}(w)^*} = \nu(L_{\mathcal{C}(w)} L_{\mathcal{C}(v)})$, showing that ν is an algebra map. That ν is an involution is clear from the definition.

The second statement can be deduced from the fact that ν commutes with the map $\alpha \mapsto \{\beta \mid D(\beta) \subset D(\alpha)\}$. \square

3. POSETS AND P -PARTITIONS

3.1. Posets and cell transfer. Let (P, \leq) be a possibly infinite poset. Let $s, t \in P$. We say that s *covers* t and write $s \succ t$ if for any $r \in P$ such that $s \geq r \geq t$ we have $r = s$ or $r = t$. The *Hasse diagram* of a poset P is the graph with vertex set equal to the elements of P and edge set equal to the set of covering relations in P . If $Q \subset P$ is a subset of the elements of P then Q has a natural induced subposet structure. If $s, t \in Q$ then $s \leq t$ in Q if and only if $s \leq t$ in P . Call a subset $Q \subset P$ *connected* if the elements in Q induce a connected subgraph in the Hasse diagram of P .

If P and Q are posets then the *disjoint sum* $P \oplus Q$ is the poset with the union $P \sqcup Q$ of elements, such that $a \leq b$ in $P \oplus Q$ if either $a \leq b \in P$ or $a \leq b \in Q$.

An *order ideal* I of P is an induced subposet of P such that if $s \in I$ and $s \geq t \in P$ then $t \in I$. A subposet $Q \subset P$ is called *convex* if for any $s, t \in Q$ and $r \in P$ satisfying $s \leq r \leq t$ we have $r \in Q$. Alternatively, a convex subposet is one which is closed under taking intervals. A convex subset Q is determined by specifying two order ideals J and I so that $J \subset I$ and $Q = \{s \in I \mid s \notin J\}$. We write $Q = I/J$. If $s \notin Q$ then we write $s < Q$ if $s < t$ for some $t \in Q$ and similarly for $s > Q$. If $s \in Q$ or s is incomparable with all elements in Q we write $s \sim Q$. Thus for any $s \in P$, exactly one of $s < Q$, $s > Q$ and $s \sim Q$ is true.

Let Q and R be two finite convex subposets of P . Define the *cell transfer operations* \wedge and \vee on the ordered pair (Q, R) by

$$(2) \quad Q \wedge R = \{s \in R \mid s < Q\} \cup \{s \in Q \mid s \sim R \text{ or } s < R\}$$

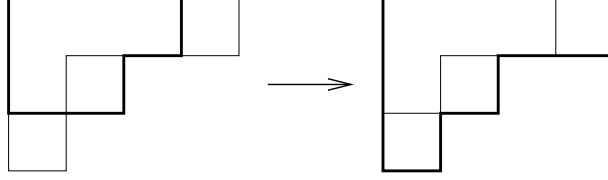


FIGURE 1. Cell transfer for the Young shape posets $\lambda = (4, 1, 1)$ and $\mu = (3, 2)$.

and

$$(3) \quad Q \vee R = \{s \in Q \mid s > R\} \cup \{s \in R \mid s \sim Q \text{ or } s > Q\}.$$

Lemma 3.1 ([LP] Lemma 3.1). *The subposets $Q \wedge R$ and $Q \vee R$ are both convex subposets of P . We have $(Q \wedge R) \cup (Q \vee R) = Q \cup R$ and $(Q \wedge R) \cap (Q \vee R) = Q \cap R$.*

The operations \vee, \wedge are not commutative, and $Q \cap R$ is a convex subposet of both $Q \vee R$ and $Q \wedge R$.

Example 3.2. The poset (\mathbb{N}^2, \leq) of (positive) points in a quadrant has cover relations $(i, j) \succ (i-1, j)$ and $(i, j) \succ (i, j-1)$. To agree with the “English” notation for Young diagrams the first coordinate i increases as we go down while the second coordinate j increases as we go to the right. The order ideals of (\mathbb{N}^2, \leq) can be identified with Young diagrams or alternatively with partitions. Let $\lambda = (4, 1, 1)$ and $\mu = (3, 2)$ be two partitions interpreted as order ideals of (\mathbb{N}^2, \leq) . Then applying the definitions (2) and (3) above one can check that $\lambda \wedge \mu = (3, 1)$ and $\lambda \vee \mu = (4, 2, 1)$. Figure 3.2 illustrates this example.

3.2. Labeled posets. Let P be a poset. A *labeling* θ of P is an injection $\theta : P \rightarrow \mathbb{P}$ into the positive integers. A descent of the labeling θ of P is a pair $p < p'$ in P such that $\theta(p) > \theta(p')$. Let us say that two labeled posets $(P, \theta_P), (Q, \theta_Q)$ are isomorphic if there is an isomorphism of posets $f : P \rightarrow Q$ so that descents are preserved. That is if $p < p'$ then $\theta(p) < \theta(p')$ if and only if $\theta'(f(p)) < \theta'(f(p'))$. We say that two labelings θ_1 and θ_2 of P are *equivalent* if the identity map on P is an isomorphism of (P, θ_1) and (P, θ_2) .

Let (P, θ) be a labeled poset. If $Q \subset P$ is a subposet, then it inherits a labeling $\theta|_Q$ by restriction. When no confusion can arise, we will often denote $\theta|_Q$ simply by θ . Note however, that the descents of $\theta|_Q$ are not completely determined by the descents of θ , unless Q is a convex subset of P .

Let (P, θ_P) and (Q, θ_Q) be labeled posets. Then the *disjoint sum* $(P \oplus Q, \theta^\oplus)$ is the labeled poset (defined up to equivalence of labelings) where θ^\oplus has the same descents as the function

$$f(a) = \begin{cases} \theta_P(a) & \text{if } a \in P, \text{ and} \\ \theta_Q(a) & \text{if } a \in Q. \end{cases}$$

Example 3.3. Let P be the diamond poset with elements $a < b, c < d$ and labeling θ_P given by $\theta_P(a) = 2, \theta_P(b) = 1, \theta_P(c) = 3$, and $\theta_P(d) = 4$. Let Q be the chain with elements $e < f < g$ and labeling θ_Q given by $\theta_Q(e) = 1, \theta_Q(f) = 3$, and $\theta_Q(g) = 2$. The one possible labeling θ^\oplus for the disjoint sum $P \oplus Q$ is given by

$\theta^\oplus(a, b, c, d, e, f, g) = 4, 3, 5, 7, 1, 6, 2$. In Figure 2, the three labeled posets (P, θ_P) , (Q, θ_Q) , and $(P \oplus Q, \theta^\oplus)$ are shown. Note that we have some freedom in choosing the labelling θ^\oplus .

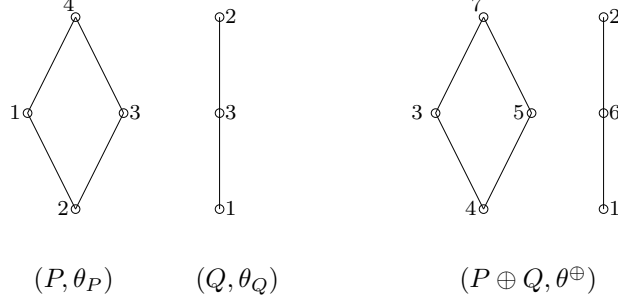


FIGURE 2

3.3. P -partitions.

Definition 3.4. A (P, θ) -partition is a map $\sigma : P \rightarrow \mathbb{P}$ such that for each covering relation $s < t$ in P we have

$$\begin{aligned} \sigma(s) &\leq \sigma(t) && \text{if } \theta(s) < \theta(t), \\ \sigma(s) &< \sigma(t) && \text{if } \theta(t) < \theta(s). \end{aligned}$$

If $\sigma : P \rightarrow \mathbb{P}$ is any map, then we say that σ *respects* θ if σ is a (P, θ) -partition.

Denote by $\mathcal{A}(P, \theta)$ the set of all (P, θ) -partitions. Clearly $\mathcal{A}(P, \theta)$ depends on (P, θ) only up to isomorphism. If P is finite then one can define the formal power series $K_{P, \theta}(x_1, x_2, \dots) \in \mathbb{Z}[[x_1, x_2, \dots]]$ by

$$K_{P, \theta}(x_1, x_2, \dots) = \sum_{\sigma \in \mathcal{A}(P, \theta)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \dots$$

The composition $\text{wt}(\sigma) = (\#\sigma^{-1}(1), \#\sigma^{-1}(2), \dots)$ is called the *weight* of σ .

Let P be a poset with n elements. Recall that a linear extension of P is a bijection $e : P \rightarrow \{1, 2, \dots, n\}$ satisfying $e(x) \leq e(y)$ if $x \leq y$ in P . The Jordan-Holder set $\mathcal{J}(P, \theta)$ of (P, θ) is the set

$$\{\theta(e^{-1}(1))\theta(e^{-1}(2)) \cdots \theta(e^{-1}(n)) \mid e \text{ is a linear extension of } P\}.$$

It is a subset of the set $\mathfrak{S}(\theta(P))$ of permutations of $\theta(P) \subset \mathbb{P}$.

Example 3.5. Suppose C is a chain $c_1 < c_2 < \dots < c_n$ with n elements and $w = w_1 w_2 \dots w_n \in \mathfrak{S}_n$ a permutation of $\{1, 2, \dots, n\}$. Then (C, w) can be considered a labeled poset, where $w(c_i) = w_i$. In this case we have $K_{C, w} = L_{C(w)}$.

Theorem 3.6 ([Sta72]). *The generating function $K_{P, \theta}$ is quasi-symmetric. We have $K_{P, \theta} = \sum_{w \in \mathcal{J}(P, \theta)} L_{D(w)}$.*

In particular, $K_{P, \theta}$ is L -positive. This motivates our treatment of $K_{P, \theta}$ as “skew”-analogues of the functions L_α . Let Q and R be two finite convex subposets of (P, θ) .

Theorem 3.7 ([LP]). *The difference $K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}$ is M -positive.*

The main theorem of [LP] generalizes Theorem 3.7 to more general labelings. We will return to a discussion of these more general labelings in Section 7.

Example 3.8. Let $P = \lambda$ be the poset of squares in the Young diagram of a partition λ as in Example 3.2. Let θ_{reading} be the labeling of λ obtained from the bottom to top row-reading order. Then $K_{\lambda, \theta_{\text{reading}}}$ is equal to the Schur function s_λ . In [LP] it is conjectured and in [LPP] it is shown that in this case the expression of Theorem 3.7 is Schur positive, which implies monomial positivity.

4. CELL TRANSFER FOR P -PARTITIONS

By Theorem 3.6, the expression $K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}$ is always a quasi-symmetric function. We now show that this difference is L -positive.

Let (P, θ) be a labeled poset and let Q and R be convex subsets. In [LP], we gave a weight preserving injection

$$\eta : \mathcal{A}(Q, \theta) \times \mathcal{A}(R, \theta) \longrightarrow \mathcal{A}(Q \wedge R, \theta) \times \mathcal{A}(Q \vee R, \theta).$$

The injection η satisfies additional crucial properties. First let us say that $i \neq j$ are *adjacent* in a multiset T (of integers) if $i, j \in T$ and for any other $t \in T$ both $i \leq t \leq j$ and $j \leq t \leq i$ fail to hold.

Proposition 4.1. *Suppose $\omega \in \mathcal{A}(Q, \theta)$ and $\sigma \in \mathcal{A}(R, \theta)$ and $\eta(\omega, \sigma) = (\omega \wedge \sigma, \omega \vee \sigma)$. Let $p \in Q \cup R$.*

- (1) *If $p \in Q \cap R$, then $\{\omega(p), \sigma(p)\} = \{\omega \wedge \sigma(p), \omega \vee \sigma(p)\}$. Furthermore, suppose $\omega(p)$ and $\sigma(p)$ are adjacent in the multiset $\omega(Q) \cup \sigma(R)$. Then $\omega \wedge \sigma(p) = \omega(p)$ and $\omega \vee \sigma(p) = \sigma(p)$.*
- (2) *If $p \in Q \wedge R$ but $p \notin Q \cap R$ then $\omega \wedge \sigma(p) = \omega(p)$ if $p \in Q$ and $\omega \wedge \sigma(p) = \sigma(p)$ if $p \in R$.*
- (3) *If $p \in Q \vee R$ but $p \notin Q \cap R$ then $\omega \vee \sigma(p) = \omega(p)$ if $p \in Q$ and $\omega \vee \sigma(p) = \sigma(p)$ if $p \in R$.*

Roughly speaking, Proposition 4.1(1) says that if $p \in Q \cap R$, then one obtains $(\omega \wedge \sigma(p), \omega \vee \sigma(p))$ by possibly “swapping” $\omega(p)$ with $\sigma(p)$; in addition, no swapping occurs if $\omega(p)$ and $\sigma(p)$ are adjacent in $\omega(Q) \cup \sigma(R)$.

Proof. Let $S \subset Q \cap R$. In [LP], $(\omega \wedge \sigma)_S : Q \wedge R \rightarrow \mathbb{P}$ was defined by

$$(\omega \wedge \sigma)_S(x) = \begin{cases} \sigma(x) & \text{if } x \in R \setminus Q \text{ or } x \in S, \\ \omega(x) & \text{otherwise,} \end{cases}$$

and $(\omega \vee \sigma)_S : Q \vee R \rightarrow \mathbb{P}$ by

$$(\omega \vee \sigma)_S(x) = \begin{cases} \omega(x) & \text{if } x \in Q \setminus R \text{ or } x \in S, \\ \sigma(x) & \text{otherwise.} \end{cases}$$

All statements except the last sentence of (1) follows from the definition of $\eta(\omega, \sigma)$ as $((\omega \wedge \sigma)_{S^\circ}, (\omega \vee \sigma)_{S^\circ})$ for some choice of $S = S^\circ$ defined in the proof of Theorem 3.7 in [LP]. The last statement of (1) follows from the fact that S° is defined to be the *smallest* set such that $((\omega \wedge \sigma)_{S^\circ}, (\omega \vee \sigma)_{S^\circ})$ is an element of $\mathcal{A}(Q \wedge R, \omega) \times \mathcal{A}(Q \vee R, \omega)$. More precisely, if $p \in Q \cap R$ is such that $\omega(p)$ and $\sigma(p)$ are adjacent then $p \notin S^\circ$. \square

Now consider the labeled posets $(Q \oplus R, \theta^\oplus)$ and $((Q \vee R) \oplus (Q \wedge R), \theta^{\vee\wedge})$, where we shall pick θ^\oplus and $\theta^{\vee\wedge}$ as follows.

For each $p \in Q \cap R$, we “duplicate” $\theta(p)$ by picking $\theta(p)' > \theta(p)$ so that for every $x \in Q \cup R$ such that $x \neq p$ we have $\theta(p)' < \theta(x)$ if and only if $\theta(p) < \theta(x)$; also the duplicates satisfying the same inequalities as the originals so that $\theta(p)' < \theta(x)'$ if and only if $\theta(p) < \theta(x)$. This describes a total order on $\{\theta(p) \mid p \in Q \cup R\} \cup \{\theta(p)' \mid p \in Q \cap R\}$. Note that we may need to replace θ with an equivalent labeling so that there is enough “space” to insert the primed letters.

Now suppose $p \in Q \cap R$. Denote the copy of p inside $Q \subset Q \oplus R$ by p_Q and the copy of p inside $R \subset Q \oplus R$ by p_R . Similarly, denote the elements of $(Q \vee R) \oplus (Q \wedge R)$. We define

$$\theta^\oplus(p) = \begin{cases} \theta(p) & \text{if } p \notin Q \cap R, \\ \theta(p)' & \text{if } p = p_Q, \\ \theta(p) & \text{if } p = p_R \end{cases}$$

and

$$\theta^{\vee\wedge}(p) = \begin{cases} \theta(p) & \text{if } p \notin Q \cap R, \\ \theta(p)' & \text{if } p = p_{Q \wedge R}, \\ \theta(p) & \text{if } p = p_{Q \vee R}. \end{cases}$$

Clearly the descents of θ^\oplus (or $\theta^{\vee\wedge}$) on either component agree with the descents of that component as a convex subposet of (P, θ) .

Theorem 4.2. *The difference $K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}$ is L -positive.*

Proof. Let $|Q| + |R| = n = |Q \vee R| + |Q \wedge R|$ and suppose $\alpha : Q \oplus R \rightarrow [n]$ is a linear extension. Then α in particular gives an element $(\alpha|_Q, \alpha|_R)$ of $\mathcal{A}(Q, \theta) \times \mathcal{A}(R, \theta)$. Using Proposition 4.1, we see that $\eta(\alpha|_Q, \alpha|_R) = (\beta|_{Q \wedge R}, \beta|_{Q \vee R})$ arises from a linear extension $\beta : (Q \wedge R) \oplus (Q \vee R) \rightarrow [n]$ (in other words the union $\beta|_{Q \wedge R} \cup \beta|_{Q \vee R}$ is exactly the interval $[n]$).

We claim that the two words

$$\begin{aligned} a_\alpha &= \theta^\oplus(\alpha^{-1}(1)) \theta^\oplus(\alpha^{-1}(2)) \dots \theta^\oplus(\alpha^{-1}(n)) \\ b_\beta &= \theta^{\vee\wedge}(\beta^{-1}(1)) \theta^{\vee\wedge}(\beta^{-1}(2)) \dots \theta^{\vee\wedge}(\beta^{-1}(n)) \end{aligned}$$

have the same descent set. Again by Proposition 4.1, the word $b = b_1 b_2 \dots b_n$ is obtained from $a = a_1 a_2 \dots a_n$ by swapping certain pairs (a_i, a_j) where $a_i = \theta^\oplus(p_Q)$ and $a_j = \theta^\oplus(p_R)$ for some $p \in Q \cap R$.

By definition $\theta^\oplus(p_Q) = \theta^{\vee\wedge}(p_{Q \wedge R})$ and $\theta^\oplus(p_R) = \theta^{\vee\wedge}(p_{Q \vee R})$ so swapping occurs if and only if $(\alpha(p_Q), \alpha(p_R)) = (\beta(p_{Q \vee R}), \beta(p_{Q \wedge R}))$. By the last statement of Proposition 4.1 (1), this never happens if $\alpha|_Q(p_Q)$ and $\alpha|_R(p_R)$ are adjacent in $[n]$, which is equivalent to $|i - j| = 1$. Thus swapping (a_i, a_j) is the same as swapping a pair of non-neighboring letters $(\theta(p), \theta(p'))$ in the word $a_1 a_2 \dots a_n$, which preserves descents by our choice of $\theta(p)'$.

We have $K_{Q, \theta} K_{R, \theta} = \sum_\alpha L_{D(a_\alpha)}$ and $K_{Q \wedge R, \theta} K_{Q \vee R, \theta} = \sum_\beta L_{D(b_\beta)}$, where the summations are over linear extensions of $Q \oplus R$ and $(Q \wedge R) \oplus (Q \vee R)$. Since η induces an injection from the first set of linear extensions to the second, we conclude that $K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}$ is L -positive. \square

Example 4.3. Let P be the poset on the 5 elements A, B, C, D, E given by the cover relations $A < B$, $A < C$, $B < D$, $B < E$, $C < D$, $C < E$. Take the following labeling θ of P : $\theta(A) = 2$, $\theta(B) = 1$, $\theta(C) = 4$, $\theta(D) = 5$, $\theta(E) = 3$. Take the

two ideals $Q = \{A, B, C, D\}$, $R = \{A, B, C, E\}$ of P . Form the disjoint sum poset $Q \oplus R$. The elements $A, B, C \in Q \cap R$ have two images in the newly formed poset: A_Q, B_Q, C_Q and A_R, B_R, C_R . The labels of $Q \oplus R$ are formed according to the rule above: for $X = A, B, C$ we have $\theta^\oplus(X_Q) = \theta(X)'$ while $\theta^\oplus(X_R) = \theta(X)$. The resulting labeling is shown in Figure 3, with θ^\oplus taking the values $\{1 < 1' < 2 < 2' < 3 < 4 < 4' < 5\}$.

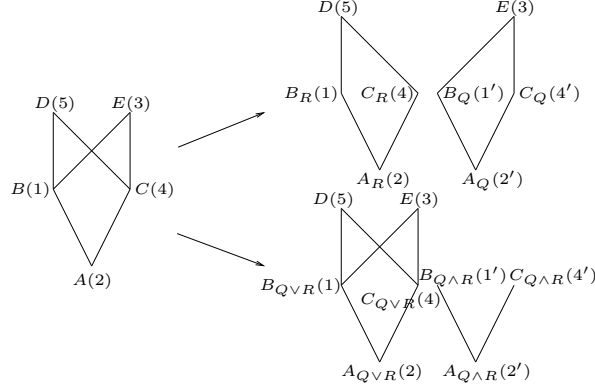


FIGURE 3. The labelings θ^\oplus of $Q \oplus R$ and $\theta^{\wedge \vee}$ of $(Q \wedge R) \oplus (Q \vee R)$ formed from a labeling θ of P . Labels are shown in parentheses.

Similarly, we obtain the labeling $\theta^{\wedge \vee}$ of $(Q \wedge R) \oplus (Q \vee R)$, as shown on Figure 3. Clearly each edge in the Hasse diagrams of $Q \oplus R$ and $(Q \wedge R) \oplus (Q \vee R)$ is a descent if and only if it is in the Hasse diagram of P .

Now, to illustrate the proof of Theorem 4.2 take a particular extension of $Q \oplus R$, namely α defined by $\alpha^{-1}([8]) = (A_Q, A_R, B_Q, C_Q, E, B_R, C_R, D)$. Performing cell transfer we get $\beta = \eta(\alpha)$ with $S^\diamond = \{B, C\}$ in the notation of the proof of Proposition 4.1, so that

$$\beta^{-1}([8]) = (A_{Q \wedge R}, A_{Q \vee R}, B_{Q \vee R}, C_{Q \vee R}, E, B_{Q \wedge R}, C_{Q \wedge R}, D)$$

In this case $a_\alpha = (2', 2, 1', 4', 3, 1, 4, 5)$ and $b_\beta = (2', 2, 1, 4, 3, 1', 4', 5)$. The pairs that got swapped are $(1, 1')$ and $(4, 4')$. Note also that the pair $(2, 2')$ did not get swapped, which we know cannot happen since those labels are neighbors in the word a_α . It is clear that the descents in a_α are indeed the same as in b_β .

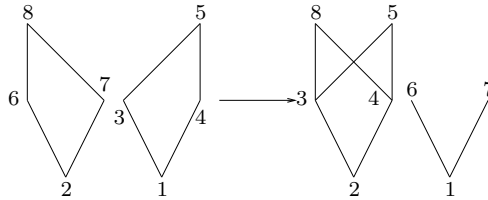


FIGURE 4. The linear extension α of $Q \oplus R$ and the linear extension β of $(Q \wedge R) \oplus (Q \vee R)$ obtained by cell transfer.

Comparing Theorem 4.2 and Theorem 3.6, we obtain the following question.

Question 4.4. When is the difference $K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}$ itself of the form $K_{S, \xi}$ for some labeled poset (S, ξ) ?

In other words, we are asking for another (hopefully natural) operation $\#$ on convex subsets Q and R of a labeled poset (P, θ) so that

$$K_{Q \# R, \theta} = K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}.$$

We will give an affirmative answer to Question 4.4 for the case of chains in Section 6. As the following example shows, the answer to Question 4.4 is not affirmative in general.

Example 4.5. Let P be the poset with four elements $\{a, b, c, d\}$ and relations $a < b, a < c, a < d$. Give P the labeling $\theta(a) = 4, \theta(b) = 1, \theta(c) = 2$, and $\theta(d) = 3$. Let Q be the ideal $\{a, b\}$ and R be the ideal $\{a, c, d\}$. Then the difference $K_{Q \wedge R, \theta} K_{Q \vee R, \theta} - K_{Q, \theta} K_{R, \theta}$ is given by

$$(4) \quad d = L_1(L_{1111} + 2L_{112} + 2L_{121} + L_{13}) - L_{11}(L_{12} + L_{111}).$$

We will argue that d is not equal to K_{S, θ_S} for any (S, θ_S) . First we claim that no term L_α in the L -expansion of d has $\alpha_1 > 1$. It is not difficult to see directly from the shuffle product that the expansion of each term in d has six L_α terms with $\alpha_1 > 1$ (in fact $\alpha_1 = 2$) and these cancel out by Theorem 4.2.

Thus using Theorem 3.6 we conclude that if $d = K_{S, \theta_S}$ then S must be a five element poset with a unique minimal element. Also one computes from (4) that S must have exactly 10 linear extensions. No poset S has these properties.

Remark 4.6. By carefully studying the cell transfer injection η of [LP], one can also give an affirmative answer to Question 4.4 for the case where P is a tree, and Q and R are order ideals so that both $Q/Q \cap R$ and $R/Q \cap R$ are connected.

Remark 4.7. Question 4.4 can be asked for the \mathbb{T} -labeled posets of [LP] and also for the differences of products of skew Schur functions studied in [LPP]. However, we will not investigate these questions in the current article.

5. CHAINS AND FUNDAMENTAL QUASI-SYMMETRIC FUNCTIONS

5.1. Cell transfer for compositions. Let (C_n, w) be the labeled chain corresponding to the permutation $w \in \mathfrak{S}_n$. Let us consider C_n to consist of the elements $\{c_1 < c_2 < \dots < c_n\}$, so that $w : C_n \rightarrow \mathbb{P}$ is given by $w(c_i) = w_i$. The convex subsets $C[i, j]$ of C_n are in bijection with intervals $[i, j] \subset [n]$.

Let $Q = [a, b]$ and $R = [c, d]$ and assume that $a \leq c$. Then we have the following two cases:

- (1) If $b \leq d$ then $Q \wedge R = Q$ and $Q \vee R = R$.
- (2) If $b \geq d$ then $Q \wedge R = [a, d]$ and $Q \vee R = [b, c]$.

Thus to obtain a non-trivial cell transfer we assume that $a < c \leq d < b$. Let $w[i, j]$ denote the word $w_i w_{i+1} \dots w_j$. Theorem 4.2 then says that the difference

$$(5) \quad L_{C(w[a, d])} L_{C(w[c, b])} - L_{C(w[a, b])} L_{C(w[c, d])}$$

is L -positive.

We now make the difference (5) more precise by translating into the language of compositions and descent sets. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_l)$ be an ordered pair of compositions. Say β can be *found inside* α if there exists a non-negative integer $m \in [0, |\alpha| - |\beta|]$ so that $D(\beta) + m$ coincides with $D(\alpha)$

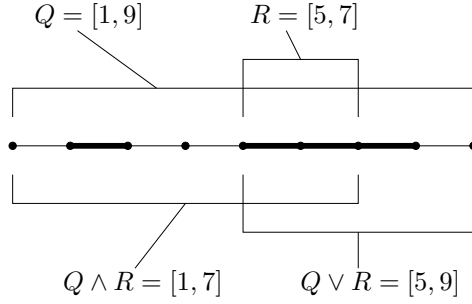


FIGURE 5. An example of the cell transfer operation for chains, here $w = (2, 1, 5, 4, 3, 7, 8, 9, 6)$, $a = 1, b = 9, c = 5, d = 7$.

restricted to $[m + 1, m + |\beta| - 1]$. We then say that β can be found inside α at m . A composition can be found inside another in many different ways. For example if $\beta = (1)$ then one may pick m to be any integer in $[0, |\alpha| - 1]$.

Now for a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \vdash n$ and an integer $x \in [1, n]$ we define two new compositions $\alpha^{x \leftarrow}, \alpha^{x \rightarrow} \vdash x$ as follows. We define $\alpha^{x \leftarrow} = (\alpha_1, \alpha_2, \dots, \alpha_{r-1}, a)$ where a, r are the unique integers satisfying $1 \leq a \leq \alpha_r$ and $\alpha_1 + \alpha_2 + \dots + \alpha_{r-1} + a = x$. Similarly, define $\alpha^{x \rightarrow} = (b, \alpha_{s+1}, \dots, \alpha_k)$ where b, s are the unique integers satisfying $1 \leq b \leq \alpha_s$ and $b + \alpha_{s+1} + \dots + \alpha_k = x$. If β can be found inside α at m , we set $\alpha \wedge_m \beta = \alpha^{(m+|\beta|) \leftarrow}$ and $\alpha \vee_m \beta = \alpha^{(|\alpha|-m) \rightarrow}$.

The L -positive expressions in (5) give the following theorem.

Theorem 5.1. *Let α and β be compositions such that β can be found inside α at m . Then the difference*

$$L_{\alpha \wedge_m \beta} L_{\alpha \vee_m \beta} - L_{\alpha} L_{\beta}$$

is L -positive.

Example 5.2. Let us take chain (C_9, w) with $w = (2, 1, 5, 4, 3, 7, 8, 9, 6)$ and $Q = [1, 9]$ and $R = [5, 7]$ so that $a = 1, b = 9, c = 5, d = 7$. Then we get the situation shown in Figure 5, the thinner edges indicating descents. If $\alpha = (1, 2, 1, 4, 1)$ and $\beta = (3)$ then there are two ways to find β inside α , and Figure 5 shows the way to find it at $m = 5$. In this case Theorem 5.1 says that $L_{(1,2,1,3)} L_{(4,1)} - L_{(1,2,1,4,1)} L_{(3)}$ is L -positive.

Remark 5.3. The operation $(\alpha, \beta) \mapsto (\alpha \wedge_m \beta, \alpha \vee_m \beta)$ interacts well with the involutions ν and ω of QSym . More precisely, if β can be found inside α at m then β^* can be found inside α^* at m and $\tilde{\beta}$ can be found inside α^* at $|\alpha| - |\beta| - m$.

5.2. The L -positivity poset. Fix a positive integer n . Now define a poset structure (PC_n, \leq) (“Pairs of Compositions”) on the set PC_n of unordered pairs $\{\alpha, \beta\}$ of compositions satisfying $|\alpha| + |\beta| = n$ by letting $\{\alpha, \beta\} \leq \{\gamma, \delta\}$ if $L_{\gamma} L_{\delta} - L_{\alpha} L_{\beta}$ is L -nonnegative. The following result relies on factorization properties of QSym which we prove in Section 8.

Proposition 5.4. *The relation $\{\alpha, \beta\} \leq \{\gamma, \delta\}$ if $L_{\gamma} L_{\delta} \geq_L L_{\alpha} L_{\beta}$ defines a partial order on the set PC_n .*

Proof. Reflexivity and transitivity of \leq are clear. Suppose we have both $\{\alpha, \beta\} \leq \{\gamma, \delta\}$ and $\{\gamma, \delta\} \leq \{\alpha, \beta\}$ then we must have $L_{\gamma} L_{\delta} = L_{\alpha} L_{\beta}$. By Corollary 8.6

and Proposition 8.11 we must have $\{\alpha, \beta\} = \{\gamma, \delta\}$. Thus \leq satisfies the symmetry condition of a partial order. \square

For an unordered pair of compositions $\{\alpha, \beta\}$ we unambiguously define another unordered pair $\{\alpha \vee \beta, \alpha \wedge \beta\}$ as follows. Suppose $|\alpha| \geq |\beta|$. If β can be found inside of α , we pick the smallest $m \in (0, |\alpha| - |\beta|)$ where this is possible and set $\alpha \wedge \beta = \alpha \wedge_m \beta$ and $\alpha \vee \beta = \alpha \vee_m \beta$. Otherwise we set $\{\alpha \vee \beta, \alpha \wedge \beta\} = \{\alpha, \beta\}$.

Conjecture 5.5. *The maximal elements of PC_n are exactly the pairs $\{\alpha, \beta\}$ for which $\{\alpha, \beta\} = \{\alpha \wedge \beta, \alpha \vee \beta\}$.*

Note that Conjecture 5.5 is compatible with the two involutions ω and ν of QSym .

Remark 5.6. (i) Conjecture 5.5 has been verified by computer up to $n = 10$.

(ii) A result similar to Conjecture 5.5 holds for the case of Schur functions: the pairs of partitions corresponding to Schur-maximal products $s_\lambda s_\mu$ are exactly those partitions fixed by “skew cell transfer”; see [LPP2].

Example 5.7. In Figure 6 the poset PC_4 is shown, a composition α being represented by a chain (C, w) satisfying $K_{C,w} = L_\alpha$. The actual labeling w of the chain is not shown, instead the descents of w are marked with thin edges. The elements of the bottom row are single compositions of size 4 since the second composition in this case is empty.

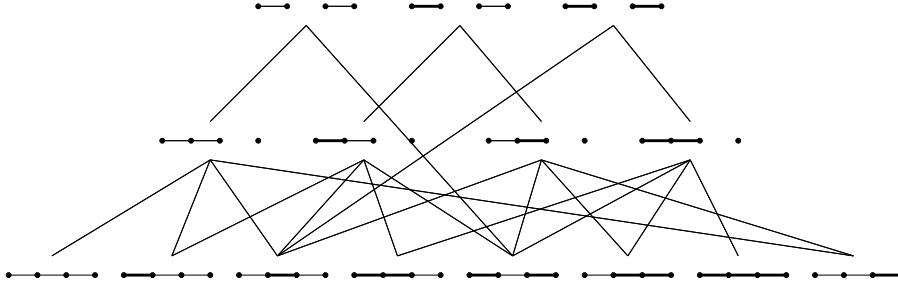


FIGURE 6. Partial order PC_4 on pairs of compositions, descents are drawn as thin edges.

One can see that maximal elements are exactly the ones for which one of the two compositions cannot be found inside the other. In this case those are exactly pairs (α, β) such that $|\alpha| = |\beta| = 2$.

6. WAVE SCHUR FUNCTIONS

In this section we define new generating functions called wave Schur functions. We first show that they are L -positive, and then prove a determinantal formula for them.

6.1. Wave Schur functions as P -partition generating functions. The poset (\mathbb{N}^2, \leq) of (positive) points in a quadrant has cover relations $(i, j) > (i - 1, j)$ and $(i, j) > (i, j - 1)$. To agree with the “English” notation for Young diagrams the first coordinate i increases as we go down while the second coordinate j increases as we go to the right. Let us fix a sequence of “strict-weak” assignments $\mathbf{p} = \{p_i \in$


$$O_{\mathbf{p}}((i, j) \succ (i-1, j)) = \overline{p_{j-i+1}} \quad \text{and} \quad O_{\mathbf{p}}((i, j) \succ (i, j-1)) = p_{j-i}.$$
$$\dots p_{-3}, p_{-2}, p_{-1}, p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, \dots =$$

In the following definition, λ/μ denotes a Young diagram $\{(i, j) \mid \mu_i \leq j \leq \lambda_i\}$ considered as a subposet of (\mathbb{N}^2, \leq) .

$$\begin{array}{ll} T(s) < T(t) & \text{if } O_{\mathbf{p}}(s < t) = \text{strict,} \\ T(s) \leq T(t) & \text{if } O_{\mathbf{p}}(s < t) = \text{weak.} \end{array}$$
$$s_{\lambda/\mu}^{\mathbf{p}}(x_1, x_2, \dots) = \sum_T x_1^{\#T^{-1}(1)} x_2^{\#T^{-1}(2)} \dots$$

The *standard* “strict-weak” assignment is given by $\mathbf{p} = \{p_i\}$ where $p_i = \text{weak}$ for all i . In this case a wave \mathbf{p} -tableau is a usual semistandard tableau, and the wave Schur function is the usual Schur function. Note, however, that in general a wave Schur function is not symmetric. However, wave Schur functions are always (P, θ) -partition generating functions.

Proposition 6.2. *Let λ/μ be a skew shape. There exists a (vertex) labeling $\theta_{\mathbf{p}} : \lambda/\mu \rightarrow \mathbb{P}$ such that $(s \leq t)$ is a descent of $\theta_{\mathbf{p}}$ if and only if $O_{\mathbf{p}}(s \leq t) = \text{strict}$. Thus $s_{\lambda/\mu}^{\mathbf{p}} = K_{\lambda/\mu, \theta_{\mathbf{p}}}$.*

Proof. We shall prove the result by induction on the number of boxes in λ/μ . Let (i, j) be any outer corner of λ/μ . In other words there are no boxes to the bottom right of (i, j) , and if we remove (i, j) from λ/μ we still obtain a valid skew shape $(\lambda/\mu)^-$. Suppose $\theta_{\mathbf{p}}^-$ has been defined for $(\lambda/\mu)^-$. If at most one of $(i-1, j)$ or $(i, j-1)$ is in $(\lambda/\mu)^-$ then one can define $\theta_{\mathbf{p}}$ by making $\theta_{\mathbf{p}}(i, j)$ either 1 or a very big value, letting $\theta_{\mathbf{p}}(i', j') = \theta_{\mathbf{p}}^-(i', j')$ for other boxes (i', j') (we may have to shift the values of $\theta_{\mathbf{p}}^-$ to be able to set $\theta_{\mathbf{p}}(i, j) = 1$).

So assume that $(i-1, j), (i, j-1) \in (\lambda/\mu)^-$. If $O_{\mathbf{p}}((i-1, j) \triangleleft (i, j)) = O_{\mathbf{p}}((i, j-1) \triangleleft (i, j))$, then $\theta_{\mathbf{p}}$ can be defined as in the previous case. So assume $O_{\mathbf{p}}((i-1, j) \triangleleft (i, j)) = \overline{O_{\mathbf{p}}((i, j-1) \triangleleft (i, j))}$. If $(i-1, j-1) \notin (\lambda/\mu)^-$ then $(\lambda/\mu)^-$ is disconnected. In this case, we may pick labelings $\theta_{\mathbf{p}}^1, \theta_{\mathbf{p}}^2$ for the two components C_1, C_2 of $(\lambda/\mu)^-$ so that we can set $\theta_{\mathbf{p}}(C_1) = \theta_{\mathbf{p}}^1(C_1) > \theta_{\mathbf{p}}(i, j) > \theta_{\mathbf{p}}(C_2) = \theta_{\mathbf{p}}^1(C_2)$.

Finally, suppose that $(i-1, j-1) \in (\lambda/\mu)^-$ and assume without loss of generality that $O_{\mathbf{p}}((i-1, j) \triangleleft (i, j)) = \text{strict} = O_{\mathbf{p}}((i-1, j-1) \triangleleft (i, j-1))$ and $O_{\mathbf{p}}((i, j-1) \triangleleft (i, j)) = \text{weak} = O_{\mathbf{p}}((i-1, j-1) \triangleleft (i-1, j))$ (we have used the definition of $O_{\mathbf{p}}$). Suppose $\theta_{\mathbf{p}}^-$ is defined. Then $\theta_{\mathbf{p}}^-(i-1, j) > \theta_{\mathbf{p}}^-(i-1, j-1) > \theta_{\mathbf{p}}^-(i, j-1)$. It suffices to define $\theta_{\mathbf{p}}(i, j)$ to be an integer very close to $\theta_{\mathbf{p}}^-(i-1, j-1)$ and $\theta_{\mathbf{p}}(i', j') = \theta_{\mathbf{p}}^-(i', j')$ for other boxes (i', j') , possibly shifting the values so that $\theta^{\mathbf{p}}(i, j)$ can be inserted. \square

Example 6.3. In Figure 8 an edge-labeling $O_{\mathbf{p}}$ of the shape $\lambda = (2, 2, 1)$ is given. Here $p_{-1} = \text{weak}$, $p_0 = \text{strict}$, $p_1 = \text{weak}$. The corresponding wave Schur function $s_{\lambda}^{\mathbf{p}}$ can be computed to be equal to $L_{(2,1,2)} + L_{(2,1,1,1)} + L_{(3,2)} + L_{(3,1,1)} + L_{(2,2,1)}$. It is easy to check that this edge-labeling does come from a vertex labeling of the underlying poset.



FIGURE 8. An edge-labeling $O_{\mathbf{p}}$ of the shape $\lambda = (2, 2, 1)$.

Remark 6.4. Proposition 6.2 implies a formula for $s_{\lambda/\mu}^{\mathbf{p}}(1, q, q^2, q^3, \dots)$ similar to that of Proposition 7.19.11 in [Sta99]. Indeed, we can consider the descent set $D_{\mathbf{p}}(T)$ of a standard tableau T with respect to $\theta_{\mathbf{p}}$. Then if we define a generalization of comajor index $\text{comaj}_{\mathbf{p}}(T) = \sum_{i \in D_{\mathbf{p}}(T)} (n - i)$, we obtain the formula

$$s_{\lambda/\mu}^{\mathbf{p}}(1, q, q^2, q^3, \dots) = \frac{\sum_T q^{\text{comaj}_{\mathbf{p}}(T)}}{(1-q)(1-q^2) \cdots (1-q^n)}.$$

However, it seems unlikely that an analog of hook content formula (see [Sta71, Theorem 15.3]) exists because the number of wave \mathbf{p} -tableaux filled with entries from 1 to n does not appear to factor nicely. In Example 6.3 the number of wave \mathbf{p} -tableaux with entries from 1 to 4 is the prime number 23.

Corollary 6.5 (Cell transfer for wave Schur functions). *Let λ/μ and ν/ρ be two skew shapes and \mathbf{p} be any “strict-weak” assignment. Then the difference $s_{\lambda/\mu \wedge \nu/\rho}^{\mathbf{p}} s_{\lambda/\mu \vee \nu/\rho}^{\mathbf{p}} - s_{\lambda/\mu}^{\mathbf{p}} s_{\nu/\rho}^{\mathbf{p}}$ is L -positive.*

Proof. Follows immediately from Theorem 4.2 and Proposition 6.2. \square

In [LPP] it is shown that the difference in Corollary 6.5 is in fact *Schur-positive* when \mathbf{p} is the standard assignment.

6.2. Jacobi-Trudi formula for wave Schur functions. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_l)$ be two partitions satisfying $\mu \subset \lambda$. Now, for each pair $1 \leq i, j \leq l$ such that $\mu_j - j + 1 < \lambda_i - i$, define the set

$$D_{ij}(\lambda, \mu) = \{\mu_j - j + 1 < a \leq \lambda_i - i \mid p_a = \text{strict}\} - (\mu_j - j + 1).$$

Set $\alpha_{ij}(\lambda, \mu) = \mathcal{C}(D_{ij}(\lambda, \mu))$ to be the corresponding composition of $\lambda_i - \mu_j - i + j$. If $\mu_j - j + 1 = \lambda_i - i$, set $\alpha_{ij}(\lambda, \mu) = (1)$. If $\mu_j - j = \lambda_i - i$ set $\alpha_{ij}(\lambda, \mu) = (0)$. Finally, if $\mu_j - j > \lambda_i - i$ set $\alpha_{ij}(\lambda, \mu) = \emptyset$. Let $L_{(0)} = 1$, $L_\emptyset = 0$.

Theorem 6.6 (Jacobi-Trudi expansion for wave Schur functions). *Let λ/μ be a skew shape. Then*

$$s_{\lambda/\mu}^{\mathbf{p}} = \det(L_{\alpha_{ij}(\lambda, \mu)})_{i,j=1}^n$$

where n is the number of rows in λ .

Example 6.7. Let $\lambda = (7, 6, 6, 4)$, $\mu = (2, 2, 1, 0)$. Then for \mathbf{p} given by

$$\dots p_{-3}, p_{-2}, p_{-1}, p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, \dots =$$

$\dots \text{strict}, \text{strict}, \text{strict}, \text{weak}, \text{strict}, \text{weak}, \text{weak}, \text{strict}, \text{strict}, \text{weak}, \text{strict}, \text{weak} \dots$

we get the shape in Figure 7, and

$$s_{\lambda/\mu}^{\mathbf{p}} = \begin{vmatrix} L_{(2,1,2)} & L_{(3,1,2)} & L_{(2,3,1,2)} & L_{(1,1,2,3,1,2)} \\ L_{(2,1)} & L_{(3,1)} & L_{(2,3,1)} & L_{(1,1,2,3,1)} \\ L_{(2)} & L_{(3)} & L_{(2,3)} & L_{(1,1,2,3)} \\ 0 & 1 & L_{(2)} & L_{(1,1,2)} \end{vmatrix}.$$

Proof of Theorem 6.6. Let us construct an oriented network $N_{\mathbf{p}}$, which depends on the choice of \mathbf{p} . Namely, we begin with the square grid built on the points in the upper half plane, with row 1 being the bottom row, and orient all edges to the right or upwards. Then we alter the all the crossings in each column C_i such that $p_i = \text{strict}$ as shown in Figure 9. Namely, we arrange these crossings so that it is impossible to move from left to right through them, but other directions that were possible before are still possible (see Figure 10). We assign to each edge in row i weight x_i , and every other edge weight 1. Now mark the points M_k with coordinates $(\mu_k - k + 1, 1)$ on our grid. Mark exit directions N_k in the columns numbered $\lambda_k - k + 1$.

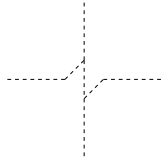


FIGURE 9. A local picture of an altered crossing.

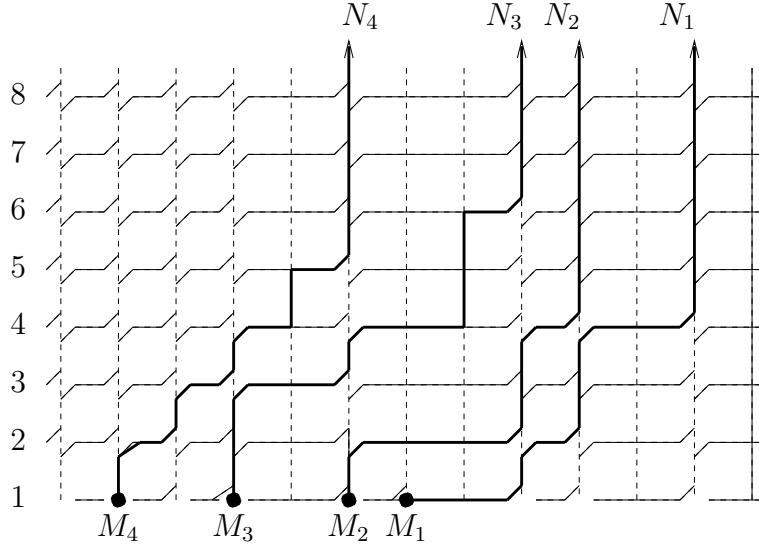


FIGURE 10. A family of paths on altered grid corresponding to \mathbf{p} -tableau on Figure 11.

Now we apply the Gessel-Viennot method to this path network; see for example [Sta99, Chapter 7] for the application of this method in the case of Schur functions. For each pair $1 \leq i, j \leq n$ the weight generating function of the paths from M_i to N_j is equal to $L_{\alpha_{ij}}(\lambda, \mu)$. Thus the determinant $\det(L_{\alpha_{ij}}(\lambda, \mu))_{i,j=1}^n$ is equal to the weight generating function of families of non-crossing paths starting at the M_i -s and ending in the columns N_i . These families of non-crossing paths are in (a weight-preserving) bijection with wave \mathbf{p} -tableau of shape λ/μ . \square

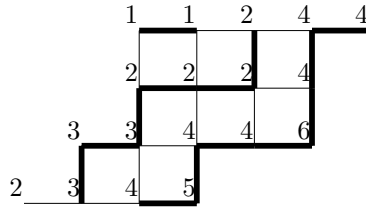


FIGURE 11. A wave \mathbf{p} -tableau of the shape $\lambda/\mu = (7, 6, 6, 4)/(2, 2, 1, 0)$ with the edge labeling $O_{\mathbf{p}}$ as in Figure 7.

Remark 6.8. (i) We have $\omega(s_{\lambda/\mu}^{\mathbf{p}}) = s_{\lambda/\mu}^{\bar{\mathbf{p}}}$ where $\bar{\mathbf{p}} = (\dots, \overline{p_{-2}}, \overline{p_{-1}}, \overline{p_0}, \overline{p_1}, \dots)$.

(ii) Let us denote $\widetilde{\lambda/\mu}$ the rotated on 180 degrees \mathbf{p} -tableau λ/μ with $\tilde{p}_i = p_{-i}$. Then $\nu(s_{\lambda/\mu}^{\mathbf{p}}) = s_{\widetilde{\lambda/\mu}}^{\bar{\mathbf{p}}}$.

The following theorem, combined with Proposition 6.2, answers Question 4.4 for the case that Q and R are convex subsets of a chain.

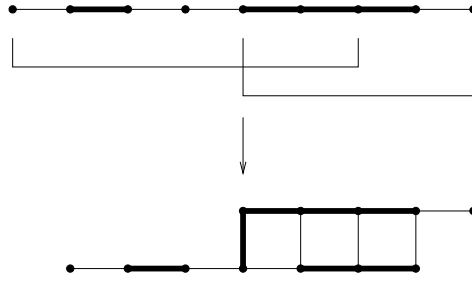


FIGURE 12. The skew shape corresponding to the difference $L_{(1,2,1,3)}L_{(4,1)} - L_{(3)}L_{(1,2,1,4,1)}$.

Theorem 6.9. *The differences $L_{\alpha \wedge_m \beta} L_{\alpha \vee_m \beta} - L_{\alpha} L_{\beta}$ of Theorem 5.1 are equal to wave Schur functions.*

Proof. We may suppose that $m \geq 1$ for otherwise the difference is 0. Pick a sequence $\mathbf{p} = \mathbf{p}^{\alpha}$ such that $p_i = \text{strict}$ if and only if $i \in D(\alpha)$ (this determines $p_1, p_2, \dots, p_{|\alpha|-1}$). Then set $\lambda = (|\alpha|, m + |\beta|)$ and $\mu = (m - 1, 1)$.

We can compute that

$$\begin{aligned} L_{\alpha_{11}}(\lambda, \mu) &= L_{\alpha \vee_m \beta} & L_{\alpha_{12}}(\lambda, \mu) &= L_{\alpha} \\ L_{\alpha_{21}}(\lambda, \mu) &= L_{\alpha \wedge_m \beta} & L_{\alpha_{22}}(\lambda, \mu) &= L_{\beta}. \end{aligned}$$

Theorem 6.6 tells us that $s_{\lambda/\mu}^{\mathbf{p}}$ is exactly $\det \begin{pmatrix} L_{\alpha \vee_m \beta} & L_{\alpha} \\ L_{\beta} & L_{\alpha \wedge_m \beta} \end{pmatrix}$.

□

We illustrate the choice of \mathbf{p}^{α} , λ and μ of Theorem 6.9 in Figure 12. Here \mathbf{p}^{α} is such that $\dots p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, \dots =$

$\dots, \text{strict}, \text{weak}, \text{strict}, \text{strict}, \text{weak}, \text{weak}, \text{weak}, \text{strict},$

$\alpha = (1, 2, 1, 4, 1), \beta = (3), m = 5$. Then $\lambda = (9, 8), \mu = (4, 1)$ and the corresponding $s_{\lambda/\mu}^{\mathbf{p}}$ is equal to $L_{(1,2,1,3)}L_{(4,1)} - L_{(3)}L_{(1,2,1,4,1)}$.

7. MORE GENERAL LABELED POSETS

Our point of view so far has been that the P -partition generating functions $K_{P,\theta}$ are “skew” analogues of the fundamental quasi-symmetric functions L_{α} , just as skew Schur functions are skew versions of the usual Schur functions. From this point of view, the two key properties that the generating functions $K_{P,\theta}$ possess are (a) they lie in QSym, and (b) they are L -positive.

7.1. \mathbb{T} -labeled posets. In [LP], we defined more general \mathbb{T} -labeled posets for which Theorem 3.7 also holds.

Let \mathbb{T} denote the set of all weakly increasing functions $f : \mathbb{P} \rightarrow \mathbb{Z} \cup \{\infty\}$. A \mathbb{T} -labeling O of a finite poset P is a map $O : \{(s, t) \in P^2 \mid s \succ t\} \rightarrow \mathbb{T}$ labeling each edge (s, t) of the Hasse diagram by a weakly increasing function $O(s, t) : \mathbb{P} \rightarrow \mathbb{Z} \cup \{\infty\}$. A \mathbb{T} -labeled poset is an ordered pair (P, O) where P is a poset, and O is a \mathbb{T} -labeling of P . It is clear how to take directed sums of \mathbb{T} -labeled posets, or to take convex subsets of \mathbb{T} -labeled posets.

A (P, O) -*tableau* is a map $\sigma : P \rightarrow \mathbb{P}$ such that for each covering relation $s \lessdot t$ in P we have $\sigma(s) \leq O(s, t)(\sigma(t))$. Note that “tableau” here is used in the same sense as “partition” was in Section 3. The weight generating function of all (P, O) -tableaux is denoted $K_{P, O}$.

Problem 7.1. For which \mathbb{T} -labeled posets is the generating function $K_{P, O}$ quasi-symmetric?

We call (P, O) *quasi-symmetric* if $K_{P, O} \in \text{QSym}$. There is a large class of quasi-symmetric \mathbb{T} -labeled posets, containing all those induced from the form (P, θ) . Define $f^{\text{weak}}(x) = x$ and $f^{\text{strict}}(x) = x - 1$. Then $K_{P, O} \in \text{QSym}$ if $O(s \lessdot t) \in \{f^{\text{weak}}, f^{\text{strict}}\}$ for every covering relation $s \lessdot t$. Following terminology of McNamara [McN], such strict-weak edge labeled posets are called *oriented*.

It is unclear how to obtain more solutions to Problem 7.1. However, we can show, using a factorization result we prove later, that the answer to Problem 7.1 is compatible with taking disjoint unions and connected components.

Proposition 7.2. *If (P, O_P) and (Q, O_Q) are quasi-symmetric then so is $(P \oplus Q, O_{P \oplus Q})$. If (P, O) is quasi-symmetric then each connected component $(P_i, O|_{P_i})$ of (P, O) is also quasi-symmetric.*

Proof. Since $K_{P \oplus Q, O_{P \oplus Q}} = K_{P, O} K_{Q, O}$, the first statement holds because QSym is a ring. The second statement holds by Theorem 8.1. \square

In particular, Proposition 7.2 says that it is not possible to obtain a quasi-symmetric \mathbb{T} -labeled poset “accidentally” by taking disjoint sums.

7.2. Oriented posets. Let us say that an orientation O of a poset P arises from a labeling θ if $O(s \lessdot t) = f^{\text{strict}}$ exactly when $(s \lessdot t)$ is a descent of θ . Clearly in this case we have $K_{P, O} = K_{P, \theta}$. Not every orientation arises from a labeling, as shown in [McN, Example 2.7]. It is also possible to find both oriented posets such that the generating function $K_{P, O}$ is L -positive and oriented posets such that $K_{P, O}$ is not L -positive; see [McN, Remark 5.9].

Problem 7.3. For which oriented posets (P, O) is the generating function $K_{P, O}$ L -positive?

It is unclear to us whether an analogue of Theorem 4.2 should hold for more general oriented posets (P, O) , such as the ones which are solutions to Problem 7.3. This would potentially expand our notion of “skew” fundamental quasi-symmetric functions beyond just $K_{P, \theta}$.

8. ALGEBRAIC PROPERTIES OF QSym

We prove in this section some algebraic results concerning QSym used earlier.

8.1. A factorization property of quasi-symmetric functions. Denote by $K = \mathbb{Z}[[x_1, x_2, x_3, \dots]]$ the ring of formal power series in infinitely many variables with bounded degree. Clearly the units in K or in $K^{(n)}$ are 1 and -1 . In this subsection, we prove the following property of QSym .

Theorem 8.1. *Suppose $f \in \text{QSym}$ and $f = \prod_i f_i$ is a factorization of f into irreducibles in K . Then $f_i \in \text{QSym}$ for each i .*

Now let $\mathbf{a} = (1 \leq a_1 < a_2 < \dots <)$ be an increasing sequence of positive integers and let A denote the set of such sequences. Define the algebra homomorphism $A_{\mathbf{a}} : K \rightarrow K$ by

$$A_{\mathbf{a}}f := f(x_{\mathbf{a}}) := f(0, \dots, 0, x_1, 0, \dots, 0, x_2, 0, \dots)$$

where x_i is placed in the a_i -th position. For a sequence \mathbf{a} , we shall also write $a(i) = a_i$ in function notation. Thus $a : \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing function.

As an example, take $a = (2, 3, 4, \dots)$, $b = (1, 3, 5, \dots)$. Then we have $A_a f = f(0, x_1, x_2, \dots)$ and $A_b \circ A_a f = A_b(A_a f) = f(0, x_1, 0, x_2, 0, x_3, \dots)$.

The following lemma is essentially the definition.

Lemma 8.2. *An element $f \in K$ is quasi-symmetric if and only if $f(x_{\mathbf{a}}) = f$ for each $\mathbf{a} \in A$.*

Let $k \geq 1$ be an integer. Define $\mathbf{a}^{(k)}$ by

$$a^{(k)}(i) = \begin{cases} i & \text{if } i < k, \\ i + 1 & \text{if } i \geq k. \end{cases}$$

Lemma 8.3. *Suppose $f \in K$ has degree n . Then f is quasi-symmetric if and only if $f(x_{\mathbf{a}}) = f$ for the sequences $\mathbf{a}^{(k)}$ for $1 \leq k \leq n$.*

Proof. The only if direction is clear. Assume that $f(x_{\mathbf{a}}) = f$ for each $\mathbf{a}^{(k)}$ for $1 \leq k \leq n$. To show that the coefficients of $x_1^{c_1} \cdots x_n^{c_n}$ and $x_{b_1}^{c_1} \cdots x_{b_n}^{c_n}$ in f are the same we use (the coefficient of $x_1^{c_1} \cdots x_n^{c_n}$ in the equality)

$$A_{\mathbf{a}^{(n)}}^{b_n - b_{n-1} - 1} \cdots A_{\mathbf{a}^{(2)}}^{b_2 - b_1 - 1} A_{\mathbf{a}^{(1)}}^{b_1 - 1} f = f.$$

□

The following lemma is a simple calculation.

Lemma 8.4. *We have $A_{\mathbf{b}} \circ A_{\mathbf{a}} = A_{\mathbf{c}}$ where $c(i) = a(b(i))$.*

Lemma 8.5. *Let $f \in K$. Suppose f has finite order with respect to $A_{\mathbf{a}}$ for every $\mathbf{a} \in A$. Then there exists $\mathbf{b} \in A$ so that $A_{\mathbf{b}}f \in \text{QSym}$.*

Proof. Given $f \in K$ invariant under $A_{\mathbf{a}^{(k)}}$ for $1 \leq k \leq t$, with t possibly 0, we will produce an $f' = A_{\mathbf{b}}f$ invariant under $A_{\mathbf{a}^{(k)}}$ for $1 \leq k \leq t + 1$. Using Lemma 8.3 and the fact that f has bounded degree this is sufficient.

So let f be invariant under $A_{\mathbf{a}^{(k)}}$ for $1 \leq k \leq t$. By assumption $A_{\mathbf{a}^{(t+1)}}$ has finite order d on f . Define $\mathbf{b} \in A$ by $b(i) = 1 + (i - 1)d$ and let $f' = A_{\mathbf{b}}f$. We claim that f' is invariant under $A_{\mathbf{a}^{(k)}}$ for $1 \leq k \leq t + 1$. We have

$$b(a^{(k)}(i)) = \begin{cases} 1 + (i - 1)d & \text{if } i < k, \\ 1 + id & \text{if } i \geq k. \end{cases}$$

In the following we will repeatedly use Lemma 8.4.

Define $\mathbf{b}^{(j)} \in A$ for $1 \leq j < k$ by

$$b^{(j)}(i) = \begin{cases} i & \text{if } i < j, \\ j + (i - j)d & \text{if } i \geq j. \end{cases}$$

Note that $A_{\mathbf{b}(j)} \circ (A_{\mathbf{a}(j)})^{d-1} = A_{\mathbf{b}(j-1)}$. Similarly define $\mathbf{c}^{(j)} \in A$ for $1 \leq j < k$ by

$$c^{(j)}(i) = \begin{cases} i & \text{if } i \leq j, \\ j + (i - j)d & j < i < k, \\ j + (i - j + 1)d & i \geq k. \end{cases}$$

Note that $A_{\mathbf{c}(j)} \circ (A_{\mathbf{a}(j)})^{d-1} = A_{\mathbf{c}(j-1)}$. We also have the three equalities

$$A_{\mathbf{c}(k-1)} = A_{\mathbf{b}(k-1)} \circ (A_{\mathbf{a}(k)})^d, \quad A_{\mathbf{b}(1)} = A_{\mathbf{b}}, \quad \text{and} \quad A_{\mathbf{c}(1)} = A_{\mathbf{a}(k)} \circ A_{\mathbf{b}}.$$

Finally using our assumptions and $1 \leq k \leq t+1$, we have

$$A_{\mathbf{b}}f = A_{\mathbf{b}(1)}f = \cdots = A_{\mathbf{b}(k-1)}f = A_{\mathbf{c}(k-1)}f = \cdots = A_{\mathbf{c}(1)}f = A_{\mathbf{a}(k)} \circ A_{\mathbf{b}}f.$$

□

Proof of Theorem 8.1. Let $\mathbf{a} \in A$. Applying $A_{\mathbf{a}}$ to $f = \prod_i f_i$ and using Lemma 8.2, we have $f = \prod_i A_{\mathbf{a}}f_i$. By Lemma 8.7 below, each $A_{\mathbf{a}}f_i$ must be equal to $\pm f_j$. In other words, $A_{\mathbf{a}}$ has finite order on each f_i and application of $A_{\mathbf{a}}$ to f_i produces (up to sign) another f_j . Using Lemma 8.5, we see that f_j must lie in QSym for some j . Now divide both sides by f_j and proceed by induction. □

Corollary 8.6. *QSym is a unique factorization domain.*

Corollary 8.6 also follows from work of Hazewinkel [Haz], who shows that QSym is a polynomial ring.

Proof. If $f \in \text{QSym}$ then two irreducible factorizations of f in QSym will also be irreducible factorizations in K , by Theorem 8.1. The theorem follows from Lemma 8.7, proven below. □

Lemma 8.7. *The ring K is a unique factorization domain.*

Proof. We start by recalling the well known fact that the polynomial rings $K^{(n)} = \mathbb{Z}[x_1, x_2, \dots, x_n]$ are unique factorization domains. An element of $f(x_1, x_2, \dots) \in K$ is determined by its images

$$f^{(n)} = f(x_1, x_2, \dots, x_n, 0, 0, \dots) \in K^{(n)}.$$

We may write $f = (f^{(n)})$ for a compatible sequence of $f^{(n)} \in K^{(n)}$.

We first claim that f is irreducible if and only if there exists $N > 0$ such that $f^{(n)}$ is irreducible for all $n > N$. Let $f = \prod f_i$ be a decomposition of f into irreducibles. Then there exists $M > 0$ so that $\deg(f_i^{(n)}) = \deg(f_i)$ for all i and $n > M$. Thus $f^{(n)} = \prod f_i^{(n)}$ is reducible for $n > M$ if f is. Conversely, suppose that $f^{(n)}$ is reducible for infinitely many values of n . If $n > M$ and $f^{(n)}$ is reducible then $f^{(m)}$ is also reducible for $n > m > M$. Thus we may assume $f^{(n)}$ is reducible for all $n > N$ for some $N > M$. Restriction of $f^{(n)}$ to $f^{(m)}$ for $n > m > N$ will not change the degree of any of the factors. Thus the factorizations of $f^{(n)}$ are compatible for each $n > N$. For sufficiently large $n \gg N$, the number k of irreducible factors of $f^{(n)}$ will be constant and greater than 1. Ordering the factorizations $\prod_{i=1}^k f_i^{(n)}$ compatibly, we conclude that $f = \prod_{i=1}^k f_i$ where $f_i = (f_i^{(n)})$ is reducible.

Now suppose that $f = \prod_i f_i = \prod_j g_j$ are two factorizations of f into irreducibles. By our claim, there exists some huge N so that $\prod_i f_i^{(n)} = \prod_j g_j^{(n)}$ are factorizations of $f^{(n)}$ into irreducibles in $K^{(n)}$, for each $n > N$. Since $K^{(n)}$ is a *UFD*, these

factorizations are the same up to permutation and sign: $g_i^{(n)} = \epsilon_i f_{\sigma(i)}^{(n)}$. If N is chosen large enough the same permutation σ and signs ϵ_i will work for all $n > N$. This shows that $g_i = \epsilon_i f_{\sigma(i)}$. \square

Remark 8.8. (i) Note that Corollary 8.6 is not true in finitely many variables. For example, in two variables x_1 and x_2 we have $(x_1^2 x_2)(x_1 x_2^2) = (x_1 x_2)^3$.

(ii) It seems interesting to ask whether the r -quasi-symmetric functions defined by Hivert [Hiv] also form a unique factorization domain. The m -quasi-invariants [EG] occurring in representation theory do not in general form unique factorization domains.

8.2. Irreducibility of fundamental quasi-symmetric functions. In this section, we show that the fundamental quasi-symmetric functions $\{L_\alpha\}$ and the monomial quasi-symmetric functions $\{M_\alpha\}$ are irreducible in QSym and in K .

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_l)$ be two compositions. Define the *lexicographic* order on compositions by $\alpha > \beta$ if and only if for some i we have $\alpha_j = \beta_j$ for $1 \leq j \leq i-1$ and $\alpha_i > \beta_i$. Using this order, we obtain lexicographic orders on monomials $\{x^\alpha\}$, monomial quasi-symmetric functions $\{M_\alpha\}$ and fundamental quasi-symmetric functions $\{L_\alpha\}$. Note that the lexicographically maximal monomial in M_α or L_α is x^α .

In the following proofs we say that a quasi-symmetric function f contains a term L_α (and similarly for M_α) if the coefficient of L_α is non-zero when f is written in the basis of fundamental quasi-symmetric functions. The following lemma is immediate from the definitions.

Lemma 8.9. *The lexicographically maximal monomial in the product fg of two quasi-symmetric functions f and g is the product of the lexicographically maximal monomials in f and g .*

Proposition 8.10. *The monomial quasi-symmetric function M_α is irreducible in QSym and in $K = \mathbb{Z}[[x_1, x_2, \dots]]$.*

Proof. We proceed by induction on the size $n = \alpha_1 + \alpha_2 + \dots + \alpha_k$. For $n = 1$ the statement is obvious.

Assume now that $M_\alpha = fg$ is not irreducible. Note first that f and g must be homogeneous. Otherwise, the homogeneous components of maximal and minimal degree in the product would not cancel out, and thus we would never get the homogeneous function M_α . Also note that according to Theorem 8.1 both f and g must be quasi-symmetric. First we suppose that $k = 1$.

Now, take the specialization $x_i = q^{i-1}$. It is known ([Sta99, Proposition 7.19.10]) that under this specialization we have $L_\alpha(1, q, q^2, \dots) = \frac{q^{e(\alpha)}}{(1-q)(1-q^2)\dots(1-q^{|\alpha|})}$, where $e(\alpha)$ is the “comajor” statistic. That means that if $\deg(f) = p$, $\deg(g) = \alpha_1 - p$ and $0 < p < \alpha_1$, fg will never have a pole at primitive α_1 -th root of unity. On the other hand $M_\alpha(1, q, \dots) = \frac{1}{1-q^{\alpha_1}}$, and thus M_α has a pole at a primitive α_1 -th root of unity, which is a contradiction.

Now suppose that $k \neq 1$. We write each of the participating functions as polynomials in x_1 :

$$M_\alpha = x_1^{\alpha_1} \tilde{M}_1 + \dots, \quad f = x_1^r \tilde{f}_1 + \dots, \quad g = x_1^{\alpha_1-r} \tilde{g}_1 + \dots.$$

Here the leading term is the one with the highest power of x_1 , and the notation \tilde{f} denotes a power series $f(x_2, x_3, \dots)$ quasi-symmetric in the variables x_2, x_3, \dots .

Note that $\tilde{M}_1 = M_{(\alpha_2, \dots, \alpha_k)}(x_2, x_3, \dots)$ is the monomial quasi-symmetric function corresponding to the composition obtained from α by removing the first part. Since $M_\alpha = f g$, we must have $M_1 = f_1 g_1$. By induction one of f_1 or g_1 is equal to a unit, ± 1 . Without loss of generality we can assume $f_1 = 1$. Thus the monomial quasi-symmetric function $M_{(r)}$ occurs in f . By Lemma 8.9 above we conclude that the lexicographically maximal monomial quasi-symmetric function in g is $M_{(\alpha_1-r, \alpha_2, \dots, \alpha_k)}$.

Now apply the involution ν of Proposition 2.1 to the equality $M_\alpha = f g$ to obtain $M_{\alpha^*} = \nu(f)\nu(g)$. By Proposition 2.1, $\nu(M_{(r)}) = M_{(r)}$ and so the monomial $M_{(r)}$ is still the lexicographically maximal monomial in $\nu(f)$. Similarly, the monomial symmetric function $M_{(\alpha_k, \dots, \alpha_2, \alpha_1-r)}$ occurs in $\nu(g)$ with non-zero coefficient. Since $k \neq 1$, the lexicographically maximal monomial in the product $\nu(f)\nu(g)$ is at least as large as $M_{(\alpha_k+r, \dots, \alpha_1-r)}$. This however is lexicographically larger than $M_{\alpha^*} = M_{(\alpha_k, \dots, \alpha_1)}$ unless $r = 0$.

We conclude that $f = 1$ and that M_α is irreducible. \square

Proposition 8.11. *The fundamental quasi-symmetric function L_α is irreducible in both QSym and in $K = \mathbb{Z}[[x_1, x_2, x_3, \dots]]$.*

Proof. The trick used for the case $k = 1$ in the proof of Proposition 8.10 also works here. \square

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